

Problem 4.31

For the most general normalized spinor χ (Equation 4.139), compute $\langle S_x \rangle$, $\langle S_y \rangle$, $\langle S_z \rangle$, $\langle S_x^2 \rangle$, $\langle S_y^2 \rangle$, and $\langle S_z^2 \rangle$. Check that $\langle S_x^2 \rangle + \langle S_y^2 \rangle + \langle S_z^2 \rangle = \langle S^2 \rangle$.

Solution

The general normalized state of a particle with spin 1/2 is

$$\chi = \begin{bmatrix} a \\ b \end{bmatrix}$$

and is subject to the condition that $|a|^2 + |b|^2 = 1$. Calculate the expectation value of S_x .

$$\begin{aligned} \langle S_x \rangle &= \frac{\langle \chi | S_x | \chi \rangle}{\langle \chi | \chi \rangle} = \frac{\chi^\dagger S_x \chi}{1} = \begin{bmatrix} a \\ b \end{bmatrix}^\dagger \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \frac{\hbar}{2} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \frac{\hbar}{2} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} \\ &= \frac{\hbar}{2} (a^* b + b^* a) \\ &= \hbar \left[\frac{a^* b + (a^* b)^*}{2} \right] \\ &= \hbar \operatorname{Re}(a^* b) \end{aligned}$$

Calculate the expectation value of S_x^2 .

$$\begin{aligned} \langle S_x^2 \rangle &= \frac{\langle \chi | S_x^2 | \chi \rangle}{\langle \chi | \chi \rangle} = \frac{\chi^\dagger S_x^2 \chi}{1} = \begin{bmatrix} a \\ b \end{bmatrix}^\dagger \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \frac{\hbar^2}{4} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \frac{\hbar^2}{4} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} \\ &= \frac{\hbar^2}{4} \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \frac{\hbar^2}{4} (a^* a + b^* b) \\ &= \frac{\hbar^2}{4} \end{aligned}$$

Calculate the expectation value of S_y .

$$\begin{aligned}
 \langle S_y \rangle &= \frac{\langle \chi | S_y | \chi \rangle}{\langle \chi | \chi \rangle} = \frac{\chi^\dagger S_y \chi}{1} = \begin{bmatrix} a \\ b \end{bmatrix}^\dagger \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\
 &= \frac{\hbar}{2} [a^* \quad b^*] \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\
 &= \frac{\hbar}{2} [a^* \quad b^*] \begin{bmatrix} -ib \\ ia \end{bmatrix} \\
 &= \frac{\hbar}{2} (-ia^*b + ib^*a) \\
 &= \hbar \left[\frac{a^*b - (a^*b)^*}{2i} \right] \\
 &= \hbar \operatorname{Im}(a^*b)
 \end{aligned}$$

Calculate the expectation value of S_y^2 .

$$\begin{aligned}
 \langle S_y^2 \rangle &= \frac{\langle \chi | S_y^2 | \chi \rangle}{\langle \chi | \chi \rangle} = \frac{\chi^\dagger S_y^2 \chi}{1} = \begin{bmatrix} a \\ b \end{bmatrix}^\dagger \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\
 &= \frac{\hbar^2}{4} [a^* \quad b^*] \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\
 &= \frac{\hbar^2}{4} [a^* \quad b^*] \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} -ib \\ ia \end{bmatrix} \\
 &= \frac{\hbar^2}{4} [a^* \quad b^*] \begin{bmatrix} a \\ b \end{bmatrix} \\
 &= \frac{\hbar^2}{4} (a^*a + b^*b) \\
 &= \frac{\hbar^2}{4} (|a|^2 + |b|^2) \\
 &= \frac{\hbar^2}{4}
 \end{aligned}$$

Calculate the expectation value of S_z .

$$\begin{aligned}
 \langle S_z \rangle &= \frac{\langle \chi | S_z | \chi \rangle}{\langle \chi | \chi \rangle} = \frac{\chi^\dagger S_z \chi}{1} = \begin{bmatrix} a \\ b \end{bmatrix}^\dagger \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\
 &= \frac{\hbar}{2} [a^* \quad b^*] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\
 &= \frac{\hbar}{2} [a^* \quad b^*] \begin{bmatrix} a \\ -b \end{bmatrix} \\
 &= \frac{\hbar}{2} (a^* a - b^* b) \\
 &= \hbar(|a|^2 - |b|^2)
 \end{aligned}$$

Calculate the expectation value of S_z^2 .

$$\begin{aligned}
 \langle S_z^2 \rangle &= \frac{\langle \chi | S_z^2 | \chi \rangle}{\langle \chi | \chi \rangle} = \frac{\chi^\dagger S_z^2 \chi}{1} = \begin{bmatrix} a \\ b \end{bmatrix}^\dagger \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\
 &= \frac{\hbar^2}{4} [a^* \quad b^*] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\
 &= \frac{\hbar^2}{4} [a^* \quad b^*] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ -b \end{bmatrix} \\
 &= \frac{\hbar^2}{4} [a^* \quad b^*] \begin{bmatrix} a \\ b \end{bmatrix} \\
 &= \frac{\hbar^2}{4} (a^* a + b^* b) \\
 &= \frac{\hbar^2}{4} (|a|^2 + |b|^2) \\
 &= \frac{\hbar^2}{4}
 \end{aligned}$$

The reason $\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle = \hbar^2/4$ is because each Pauli matrix is its own inverse.

$$\begin{aligned}
 \langle S_j^2 \rangle &= \langle \chi | S_j^2 | \chi \rangle = \left\langle \chi \left| \frac{\hbar}{2} \sigma_j \frac{\hbar}{2} \sigma_j \right| \chi \right\rangle = \frac{\hbar^2}{4} \langle \chi | \sigma_j \sigma_j^{-1} | \chi \rangle = \frac{\hbar^2}{4} \langle \chi | 1 | \chi \rangle = \frac{\hbar^2}{4} \langle \chi | \chi \rangle \\
 &= \frac{\hbar^2}{4} (1) \\
 &= \frac{\hbar^2}{4}
 \end{aligned}$$

The defining eigenvalue problem for S^2 is in Equation 4.135 on page 166.

$$S^2|s m_s\rangle = \hbar^2 s(s+1)|s m_s\rangle$$

Premultiply both sides by $\langle s m_s|$.

$$\langle s m_s| S^2 |s m_s\rangle = \langle s m_s| \hbar^2 s(s+1) |s m_s\rangle$$

$$\langle s m_s| S^2 |s m_s\rangle = \hbar^2 s(s+1) \langle s m_s| s m_s\rangle$$

$$\frac{\langle s m_s| S^2 |s m_s\rangle}{\langle s m_s| s m_s\rangle} = \hbar^2 s(s+1)$$

$$\langle S^2 \rangle = \hbar^2 s(s+1)$$

For spin 1/2,

$$\langle S^2 \rangle = \hbar^2 \frac{1}{2} \left(\frac{1}{2} + 1 \right) = \frac{3\hbar^2}{4},$$

so

$$\langle S_x^2 \rangle + \langle S_y^2 \rangle + \langle S_z^2 \rangle = \frac{\hbar^2}{4} + \frac{\hbar^2}{4} + \frac{\hbar^2}{4} = \frac{3\hbar^2}{4} = \langle S^2 \rangle.$$